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Q No (2) If $\cosh x = \sec\theta$, prove that

(i) $\tanh^2\frac{x}{2} = \tan^2\frac{\theta}{2}$

(ii) $x = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Solution (i) Given $\cosh x = \sec\theta$

or $\cosh x = \frac{1}{\cos\theta}$

$$\frac{\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x}{\cosh^2\frac{1}{2}x - \sinh^2\frac{1}{2}x} = \frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$\left[\begin{array}{l}\cosh^2\frac{1}{2}x - \sinh^2\frac{1}{2}x = 1 \\ \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1\end{array}\right]$$

Dividing each term in the num. and denom. of L.H.S by $\cosh^2\frac{1}{2}x$ and R.H.S by $\cos^2\frac{\theta}{2}$

$$\frac{\dfrac{\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x}{\cosh^2\frac{1}{2}x}}{\dfrac{\cosh^2\frac{1}{2}x - \sinh^2\frac{1}{2}x}{\cosh^2\frac{1}{2}x}} = \frac{\dfrac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}$$

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$$\text{or}\quad \dfrac{\dfrac{\cancel{\cosh^2 \frac{1}{2}x}}{\cancel{\cosh^2 \frac{1}{2}x}} + \dfrac{\sinh^2 \frac{1}{2}x}{\cosh^2 \frac{1}{2}x}}{\dfrac{\cancel{\cosh^2 \frac{1}{2}x}}{\cancel{\cosh^2 \frac{1}{2}x}} - \dfrac{\sinh^2 \frac{1}{2}x}{\cosh^2 \frac{1}{2}x}} = \dfrac{\dfrac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} + \dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} - \dfrac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}$$

$$\text{or}\quad \frac{1+\tanh^2\frac{1}{2}x}{1-\tanh^2\frac{1}{2}x} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

Applying Componendo and dividendo

we get

$$\frac{(1+\tanh^2\frac{1}{2}x)+(1-\tanh^2\frac{1}{2}x)}{(1+\tanh^2\frac{1}{2}x)-(1-\tanh^2\frac{1}{2}x)} = \frac{1+\tan^2\frac{\theta}{2}+(1-\tan^2\frac{\theta}{2})}{1+\tan^2\frac{\theta}{2}-(1-\tan^2\frac{\theta}{2})}$$

$$\text{or}\quad \frac{(1+\cancel{\tanh^2\frac{1}{2}x}+1-\cancel{\tanh^2\frac{1}{2}x})}{(\cancel{1}+\tanh^2\frac{1}{2}x-\cancel{1}+\tanh^2\frac{1}{2}x)} = \frac{1+\cancel{\tan^2\frac{\theta}{2}}+1-\cancel{\tan^2\frac{\theta}{2}}}{\cancel{1}+\tan^2\frac{\theta}{2}-\cancel{1}+\tan^2\frac{\theta}{2}}$$

$$\text{or}\quad \frac{\cancel{2}}{\cancel{2}\tanh^2\frac{1}{2}x} = \frac{\cancel{2}}{\cancel{2}\tan^2\frac{\theta}{2}}$$

$$\text{or}\quad \tanh^2\frac{1}{2}x = \tan^2\frac{\theta}{2}$$

proved

(11) Give $\cosh x = \sec\theta$

or $\frac{e^x + e^{-x}}{2} = \frac{1}{\cos\theta}$

Applying componendo and dividendo, we get

$$\frac{e^x + e^{-x} + 2}{e^x + e^{-x} - 2} = \frac{1+\cos\theta}{1-\cos\theta}$$

or $$\frac{e^x + e^{-x} + 2 \times e^x \times e^{-x}}{e^x + e^{-x} - 2 \times e^x \times e^{-x}} = \frac{1+\cos\theta}{1-\cos\theta}$$

$$\frac{(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2}{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2} = \frac{\cancel{2}\cos^2\frac{\theta}{2}}{\cancel{2}\sin^2\frac{\theta}{2}}$$

or $$\frac{(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2}{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2} = \cot^2\frac{\theta}{2}$$

or $$\frac{(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2}{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2} = \frac{1}{\tan^2\frac{\theta}{2}}$$

or $$\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \frac{1}{\tan\frac{\theta}{2}}$$

Applying componendo and dividendo, we get

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$$\frac{\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)-\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)}=\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}$$

or $$\frac{e^{\frac{x}{2}}+\cancel{e^{-x/2}}+e^{\frac{x}{2}}-\cancel{e^{-x/2}}}{\cancel{e^{x/2}}+e^{-\frac{x}{2}}-\cancel{e^{x/2}}+e^{-\frac{x}{2}}}=\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}$$

or $$\frac{\cancel{2}e^{\frac{x}{2}}}{\cancel{2}e^{-\frac{x}{2}}}=\frac{\tan\frac{\pi}{4}+\tan\frac{1}{2}\theta}{1-\tan\frac{\pi}{4}\tan\frac{1}{2}\theta}$$

$$\frac{e^{\frac{x}{2}}}{e^{-\frac{x}{2}}}=\frac{\tan\frac{\pi}{4}+\tan\frac{1}{2}\theta}{1-\tan\frac{\pi}{4}\tan\frac{1}{2}\theta}$$

$$e^{\frac{x}{2}}\times e^{\frac{x}{2}}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)$$

$$e^{\frac{x+x}{2}}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)$$

$$e^{\frac{\cancel{2}x}{\cancel{2}}}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)$$

$$e^{x}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)$$

or $x=\log\tan\left(\frac{\pi}{4}+\frac{1}{2}\theta\right)$ proved

Q No (3) If $\sin(A+iB) = x+iy$, prove that

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

Solution

$$x+iy = \sin(A+iB)$$

or $$x+iy = \sin A \cos iB + \cos A \sin iB$$

or $$x+iy = \sin A \cosh B + i \cos A \sinh B$$

$$\left[\begin{array}{l}\cos iB = \cosh B \\ \sin iB = i \sinh B\end{array}\right.$$

Equating real and imaginary parts on both side, we have

$$x = \sin A \cosh B \quad --\ (I)$$

and $$y = \cos A \sinh B \quad --\ (II)$$

From (I) and (II) we get

$$\sin A = \frac{x}{\cosh B} \text{ and } \cos A = \frac{y}{\sinh B}$$

$$\therefore \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$
$$= 1$$

Hence proved

Q.No(4) If $x+iy=\cosh(u+iv)$ prove that

(a) $\dfrac{x^2}{\cos^2 v}-\dfrac{y^2}{\sin^2 v}=1$

and (b) $\dfrac{x^2}{\cosh^2 u}+\dfrac{y^2}{\sinh^2 u}=1$

Solution $x+iy=\cosh(u+iv)$

$=\cos[i(u+iv)]$

or $x+iy=\cos[iu+i^2v]$

or $x+iy=\cos[iu-v]$

$[i^2=-1]$

or $x+iy=\cos(iu)\cos v+\sin(iu)\sin v$

or $x+iy=\cosh u\cos v+i\sinh u\sin v$

Equaling real and imaginary parts on both sides, we get

$x=\cosh u\cos v$ --- (I)

and $y=\sinh u\sin v$ --- (II)

from (I) and (II)

$\cosh u=\dfrac{x}{\cos v}$ and $\sinh u=\dfrac{y}{\sin v}$

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Squaring and subtracting, we get

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u$$
$$= 1$$

Hence proved

(b) From (I) and (II) we get

$$\cos v = \frac{x}{\cosh u} \text{ and } \sin v = \frac{y}{(\sinh u)}$$

Squaring and adding, we get

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v$$
$$= 1$$

Hence proved