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Q(4) If $\cos(\theta+i\phi)=R(\cos\alpha+i\sin\alpha)$, prove that

$$\phi=\frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

Solution. Given that

$$R(\cos\alpha+i\sin\alpha)=\cos(\theta+i\phi)$$
$$=\cos\theta\cdot\cos(i\phi)-\sin\theta\sin(i\phi)$$

or $R\cos\alpha+Ri\sin\alpha=\cos\theta\cos(i\phi)-\sin\theta\sin(i\phi)$

or $R\cos\alpha+Ri\sin\alpha=\cos\theta\cosh\phi-i\sin\theta\sinh\phi$

$$\left[\begin{array}{l}\cos i\phi=\cosh\phi\\ \sin(i\phi)=i\sinh\phi\end{array}\right]$$

Equating real and imaginary parts, we have

$$R\cos\alpha=\cos\theta\cosh\phi \quad \text{—— (I)}$$

$$R\sin\alpha=-\sin\theta\sinh\phi \quad \text{—— (II)}$$

Dividing (II) by (I), we get

$$\frac{R\sin\alpha}{R\cos\alpha}=-\frac{\sin\theta\sinh\phi}{\cos\theta\cosh\phi}$$

$$\tan\alpha=-\frac{\sin\theta\sinh\phi}{\cos\theta\cosh\phi}$$

$$\frac{\sin\alpha}{\cos\alpha} = -\frac{\sin\theta \sinh\phi}{\cos\theta \cosh\phi}$$

or $$\tan\alpha = -\tan\theta \tanh\phi$$

or $$\frac{\tan\alpha}{\tan\theta} = -\tanh\phi$$

or $$\frac{\frac{\sin\alpha}{\cos\alpha}}{\frac{\sin\theta}{\cos\theta}} = -\frac{\sinh\phi}{\cosh\phi}$$

or $$\frac{\sin\alpha}{\cos\alpha} \times \frac{\cos\theta}{\sin\theta} = -\frac{\sinh\phi}{\cosh\phi}$$

or $$\frac{\sin\alpha \cos\theta}{\cos\alpha \sin\theta} = -\frac{\sinh\phi}{\cosh\phi}$$

or $$\frac{\sin\alpha \cos\theta}{\cos\alpha \sin\theta} = -\frac{\frac{(e^{\phi} + e^{-\phi})}{2}}{\frac{e^{\phi} + e^{-\phi}}{2}}$$

or $$\frac{\sin\alpha \cos\theta}{\cos\alpha \sin\theta} = -\frac{(e^{\phi} - e^{-\phi})}{\cancel{2}} \times \frac{\cancel{2}}{(e^{\phi} + e^{-\phi})}$$

or $$\frac{\sin\alpha \cos\theta}{\cos\alpha \sin\theta} = -\frac{(e^{\phi} - e^{-\phi})}{(e^{\phi} + e^{-\phi})}$$

or $\dfrac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta} = \dfrac{-e^{\phi}+e^{-\phi}}{e^{\phi}+e^{-\phi}}$

$= \dfrac{e^{-\phi}-e^{\phi}}{e^{\phi}+e^{-\phi}}$

or $\dfrac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta} = \dfrac{e^{-\phi}-e^{\phi}}{e^{\phi}+e^{-\phi}}$

Applying componendo and dividendo, we

$\dfrac{\cos\alpha\sin\theta-\sin\alpha\cos\theta}{\cos\alpha\sin\theta+\sin\alpha\cos\theta} = \dfrac{(e^{\phi}+e^{-\phi})-(e^{-\phi}-e^{\phi})}{(e^{\phi}+e^{-\phi})+(e^{-\phi}-e^{\phi})}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \dfrac{e^{\phi}+\cancel{e^{-\phi}}-\cancel{e^{-\phi}}+e^{\phi}}{\cancel{e^{\phi}}+e^{-\phi}+e^{-\phi}-\cancel{e^{\phi}}}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \dfrac{\cancel{2}e^{\phi}}{\cancel{2}e^{-\phi}}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \dfrac{e^{\phi}}{e^{-\phi}}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = e^{\phi}\times e^{\phi}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = e^{\phi+\phi}$

or $\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = e^{2\phi}$

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or $2\phi = \log \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$

or $\phi = \frac{1}{2} \log \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$

Hence proved

Q.No(5) If $\sin(\theta+i\phi) = \cos\alpha + i\sin\alpha$, prove that

$\sin\alpha = \pm\cos^2\theta = \pm\sinh^2\phi$

Solution

$\cos\alpha + i\sin\alpha = \sin(\theta+i\phi)$

or $\cos\alpha + i\sin\alpha = \sin\theta\cos(i\phi) + \cos\theta\sin(i\phi)$

or $\cos\alpha + i\sin\alpha = \sin\theta\cosh\phi + i\cos\theta\sinh\phi$

Equating real and imaginary parts on both sides, we have

$\cos\alpha = \sin\theta\cosh\phi$ ——— (I)

and $\sin\alpha = \cos\theta\sinh\phi$ ——— (II)

from (I)

$\sin\theta = \frac{\cos\alpha}{\cosh\phi}$

and from (II) $\cos\theta = \frac{\sin\alpha}{\sinh\phi}$

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$\therefore \cos^2\theta + \sin^2\theta = \frac{\sin^2\alpha}{\sinh^2\phi} + \frac{\cos^2\alpha}{\cosh^2\phi}$

or $1 = \frac{\sin^2\alpha \cosh^2\phi + \cos^2\alpha \sinh^2\phi}{\sinh^2\phi \cosh^2\phi}$

or $\cosh^2\phi \sinh^2\phi = \sin^2\alpha \cosh^2\phi + \cos^2\alpha \sinh^2\phi$

or $(1+\sinh^2\phi)\sinh^2\phi = \sin^2\alpha(1+\sinh^2\phi) + \cos^2\alpha \sinh^2\phi$

or $\sinh^2\phi + \sinh^4\phi = \sin^2\alpha + \sin^2\alpha \sinh^2\phi + \cos^2\alpha \sinh^2\phi$

or $\sinh^2\phi + \sinh^4\phi = \sin^2\alpha + \sinh^2\phi(\sin^2\alpha + \cos^2\alpha)$

or $\sinh^2\phi + \sinh^4\phi = \sin^2\alpha + \sinh^2\phi$

or $\cancel{\sinh^2\phi} + \sinh^4\phi - \cancel{\sinh^2\phi} = \sin^2\alpha$

or $\sinh^4\phi = \sin^2\alpha$

or Taking square root

or $\pm \sinh^2\phi = \sin\alpha$

Hence proved

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from (i) and (ii) we get

$\cosh\phi = \frac{\cos\alpha}{\sin\theta}$ and $\sinh\phi = \frac{\sin\alpha}{\cos\theta}$

$\therefore \cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$

or $1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$

or $\cos^2\theta\cos^2\alpha - \sin^2\alpha\sin^2\theta = \sin^2\theta\cos^2\theta$

or $\cos^2\alpha\cos^2\theta - \sin^2\alpha(1-\cos^2\theta) = \cos^2\theta(1-\cos^2\theta)$

or $\cos^2\alpha\cos^2\theta - \sin^2\alpha + \sin^2\alpha\cos^2\theta = \cos^2\theta - \cos^4\theta$

or $\cos^2\alpha\cos^2\theta + \sin^2\alpha\cos^2\theta - \sin^2\alpha = \cos^2\theta - \cos^4\theta$

or $\cos^2\theta(\cos^2\alpha + \sin^2\alpha) - \sin^2\alpha = \cos^2\theta - \cos^4\theta$

or $\cos^2\theta - \sin^2\alpha = \cos^2\theta - \cos^4\theta$

or $\cos^2\theta - \cos^2\theta - \sin^2\alpha = -\cos^4\theta$

or $-\sin^2\alpha = -\cos^4\theta$

or $\sin^2\alpha = \cos^4\theta$

Taking square root

$\sin\alpha = \pm\cos^2\theta$ Hence proved

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V.V.I

Q.No.⑥ If $\tan(\theta+i\phi) = \sin(x+iy)$, then
$\coth y \sinh 2\phi = \cot x \sin 2\theta$

Solution

Given that $\tan(\theta+i\phi) = \sin(x+iy)$

or $\dfrac{\sin(\theta+i\phi)}{\cos(\theta+i\phi)} = \sin x \cos(iy) + \cos x \sin(iy)$

or $\dfrac{2\sin(\theta+i\phi)\times\cos(\theta-i\phi)}{2\cos(\theta+i\phi)\times\cos(\theta-i\phi)} = \sin x\cos(iy) + \cos x \sin(iy)$

[multiplying num and denom by $2\cos(\theta-i\phi)$]

or $\dfrac{\sin[(\theta+i\phi)+(\theta-i\phi)] + \sin[(\theta+i\phi)-(\theta-i\phi)]}{\cos[(\theta+i\phi)+(\theta-i\phi)] + \cos[(\theta+i\phi)-(\theta-i\phi)]}$

$= \sin x \cos(iy) + \cos x \sin(iy)$

or $\dfrac{\sin[\theta+\cancel{i\phi}+\theta-\cancel{i\phi}] + \sin[\cancel{\theta}+i\phi-\cancel{\theta}+i\phi]}{\cos[\theta+\cancel{i\phi}+\theta-\cancel{i\phi}] + \cos[\cancel{\theta}+i\phi-\cancel{\theta}+i\phi]}$

$= \sin x \cos(iy) + \cos x \sin(iy)$

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or $\frac{\sin 2\theta + \sin(2i\phi)}{\cos(2\theta) + \cos(2i\phi)} = \sin x \cosh y + i \cos x \sinh y$

Equating real and Imaginary parts on both sides, we get

or $\frac{\sin 2\theta}{\cos(2\theta) + \cos(2i\phi)} + \frac{i \sinh 2\phi}{\cos 2\theta + \cos(2i\phi)}$

$= \sin x \cosh y + i \cos x \sinh y$

$\frac{\sin 2\theta}{\cos 2\theta + \cos(2i\phi)} + \frac{i \sinh 2\phi}{\cos 2\theta + \cos(2i\phi)}$

$= \sin x \cosh y + i \cos x \sinh y$

$\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y$ —— (1)

$\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \cos x \sinh y$ —— (11)

Dividing (1) and (11) we get

$$\frac{\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi}}{\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}} = \frac{\sin x \cosh y}{\cos x \sinh y}$$

or $\frac{\sin 2\theta}{\cancel{\cos 2\theta + \cosh 2\phi}} \times \frac{\cancel{\cos 2\theta + \cosh 2\phi}}{\sinh 2\phi} = \frac{\sin x \cosh y}{\cos x \sinh y}$

or $\frac{\sin 2\theta}{\sinh 2\phi} = \frac{\sin x \cosh y}{\cos x \sinh y}$

or $\frac{\sin 2\theta}{\sinh 2\phi} = \tan x \coth y$

or $\frac{\sin 2\theta}{\sinh 2\phi} = \frac{\coth y}{\cot x}$

or $\sin 2\theta \cot x = \sinh 2\phi \cot x$

Hence proved

Q No (7) If $\tan(x+iy) = \cosh(u+iv)$, prove that $\tanh u \tan v = \operatorname{cosec} 2x \sinh 2y$

Solution Given that

$\tan(x+iy) = \cosh(u+iv)$

or $\frac{\sin(x+iy)}{\cos(x+iy)} = \cos[(u+iv)i]$

or $\frac{\sin(x+iy)}{\cos(x+iy)} = \cos[ui + i^2 v]$

or $\frac{\sin(x+iy)}{\cos(x+iy)} = \cos(ui - v)$

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$$\Rightarrow \frac{2\sin(x+iy)\times\cos(x-iy)}{2\cos(x+iy)\times\cos(x-iy)} = \cos(ui-v)$$

(multiplying num and denom by $2\cos(x-iy)$)

$$\Rightarrow \frac{\sin[(x+iy)+(x-iy)]+\sin[(x+iy)-(x-iy)]}{\cos[(x+iy)+(x-iy)]+\cos[(x+iy)-(x-iy)]} = \cos(ui-v)$$

$[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)$
and $2\cos A\cos B = \cos(A+B)+\cos(A-B)]$

$$\Rightarrow \frac{\sin[x+\cancel{iy}+x-\cancel{iy}]+\sin[\cancel{x}+iy-\cancel{x}+iy]}{\cos[x+\cancel{iy}+x-\cancel{iy}]+\cos[\cancel{x}+iy-\cancel{x}+iy]} = \cos(ui-v)$$

$$\Rightarrow \frac{\sin(2x)+\sin(2iy)}{\cos(2x)+\cos(2iy)} = \cos(ui-v)$$

$$\Rightarrow \frac{\sin(2x)+i\sinh 2y}{\cos 2x+\cosh 2y} = \cos(ui)\cos v+\frac{\sin(ui)}{\sin v}$$

$$\Rightarrow \frac{\sin 2x}{\cos 2x+\cosh 2y}+i\frac{\sinh 2y}{\cos 2x+\cosh 2y} = \cosh u\cos v + i\frac{\sinh u}{\sin v}$$

Equating real and imaginary parts on both sides we have

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \cosh u \cos v \quad \text{(i)}$$

and

$$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \sinh u \sin v \quad \text{(ii)}$$

Dividing (i) by (ii) we have

$$\frac{\dfrac{\sin 2x}{\cos 2x + \cosh 2y}}{\dfrac{\sinh 2y}{\cos 2x + \cosh 2y}} = \frac{\cosh u \cos v}{\sinh u \sin v}$$

or

$$\frac{\sin 2x}{\cancel{\cos 2x + \cosh 2y}} \times \frac{\cancel{\cos 2x + \cosh 2y}}{\sinh 2y} = \frac{\cosh u \cos v}{\sinh u \sin v}$$

&

$$\frac{\sin 2x}{\sinh 2y} = \frac{\cosh u}{\sinh u} \cdot \frac{\cos v}{\sin v}$$

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$$\Rightarrow \frac{\sin 2y}{\sinh 2y} = \coth u \cot v$$

$$\Rightarrow \frac{1}{\operatorname{cosec} 2x} \times \frac{1}{\sinh 2y} = \coth u \cot v$$

$$\Rightarrow \frac{1}{\operatorname{cosec} 2x} \times \frac{1}{\sinh 2y} = \frac{1}{\tanh u} \times \frac{1}{\tan v}$$

$$\Rightarrow \tanh u \tan v = \operatorname{cosec} 2x \sinh 2y$$

Hence proved