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Q.No ① Separate into real and imaginary parts

ⓐ $\sin(\alpha + i\beta)$

Solution.

$$\sin(\alpha+i\beta)$$
$$= \sin\alpha \cos(i\beta) + \cos\alpha \sin(i\beta)$$
$$= \sin\alpha \cosh\beta + \cos\alpha\,(i\sinh\beta)$$
$$= \sin\alpha \cosh\beta + i\cos\alpha \sinh\beta$$

$\because \sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$

(b) $\sinh(\alpha+i\beta)$

Solution

$$\sinh(\alpha+i\beta)$$
$$= \frac{1}{i}\sin[i(\alpha+i\beta)]$$

$[\because \sin(i\theta) = i\sinh\theta]$

$$= \frac{i}{i^2}[\sin(i\alpha + i^2\beta)]$$
$$= \frac{i}{i^2}[\sin(i\alpha - \beta)]$$

$[\because i^2 = -1]$

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$$= \frac{i}{-1}\left[\sin(i\alpha-\beta)\right]$$

$$= -i\left[\sin(i\alpha-\beta)\right]$$

$$= -i\left[\sin(i\alpha)\cos\beta-\cos(i\alpha)\sin\beta\right]$$

$$= -i\left[i\sinh\alpha\cos\beta-\cosh\alpha\sin\beta\right]$$

$$= -i\times i\sinh\alpha\cos\beta+i\cosh\alpha\sin\beta$$

$$= -i^2\sinh\alpha\cos\beta+i\cosh\alpha\sin\beta$$

$$= -\times-1\sinh\alpha\cos\beta+i\cosh\alpha\sin\beta$$

$\left[\sin(i\theta) = i\sinh\theta \text{ and } \cos(i\theta) = \cosh\theta\right]$

$$= \sinh\alpha\cos\beta+i\cosh\alpha\sin\beta$$

(c) $\tanh(\alpha+\beta i)$

Solution $\tanh(\alpha+i\beta)$

$$= \frac{\sinh(\alpha+i\beta)}{\cosh(\alpha+i\beta)}$$

$$= \frac{1}{i}\,\frac{\sin i(\alpha+i\beta)}{\cos i(\alpha+i\beta)}$$

$\left[\because \sin(i\theta) = i\sinh\theta,\ \cos(i\theta) = \cosh\theta\right]$

$$= \frac{1}{i}\,\frac{\sin(i\alpha+i^2\beta)}{\cos(i\alpha+i^2\beta)}$$

$\left[i^2 = -1\right]$

$$= \frac{1}{i}\,\frac{\sin(i\alpha-\beta)}{\cos(i\alpha-\beta)}$$

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$$= \frac{1}{i} \cdot \frac{2\sin(i\alpha-\beta)\cos(i\alpha+\beta)}{2\cos(i\alpha-\beta)\cdot\cos(i\alpha+\beta)}$$

[multiplying num and denom by $2\cos(i\alpha+\beta)$]

$$= \frac{1}{i} \cdot \frac{\sin[(i\alpha-\beta)+(i\alpha+\beta)] + \sin[(i\alpha-\beta)-(i\alpha+\beta)]}{\cos[(i\alpha-\beta)+(i\alpha+\beta)] + \cos[(i\alpha-\beta)-(i\alpha+\beta)]}$$

$$= \frac{1}{i} \cdot \frac{\sin[i\alpha-\cancel{\beta}+i\alpha+\cancel{\beta}] + \sin[\cancel{i\alpha}-\beta-\cancel{i\alpha}-\beta]}{\cos[i\alpha-\cancel{\beta}+i\alpha+\cancel{\beta}] + \cos[\cancel{i\alpha}-\beta-\cancel{i\alpha}-\beta]}$$

$$= \frac{1}{i} \cdot \frac{\sin[2i\alpha] + \sin[(-2\beta)]}{\cos[2i\alpha] + \cos(-2\beta)]}$$

$$= \frac{1}{i} \cdot \frac{\sin(2i\alpha) - \sin(2\beta)}{\cos(2i\alpha) + \cos(2\beta)}$$

[$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$]

$$= \frac{1}{i}\left[\frac{i\sinh 2\alpha - \sin 2\beta}{\cosh 2\alpha + \cos 2\beta}\right]$$

[$\sin(i\theta) = i\sinh\theta$
$\cos(i\theta) = \cosh\theta$]

$$= \frac{1}{i}\left[\frac{i\sinh 2\alpha + i^2 \sin 2\beta}{\cosh 2\alpha + \cos 2\beta}\right]$$

[$i^2 = -1$]

$$= \frac{1}{\cancel{i}} \times \frac{\cancel{i}\sinh 2\alpha}{\cosh 2\alpha + \cos 2\beta} + \frac{\cancel{i}^2}{\cancel{i}} \frac{\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$$

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$$= \frac{\sinh 2\alpha}{\cosh 2\alpha + \cos 2\beta} + i\frac{\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$$

v.v.I
Q(8)

If $u = \log_e \tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$

prove that

(a) $\tanh\frac{1}{2}u = \tan\frac{1}{2}\theta$ (b) $\cosh u = \sec\theta$

(c) $\sinh u = \tan\theta$ (d) $\tanh u = \sin\theta$

Solution $u = \log_e \tan\left(\frac{1}{4}\pi + \cancel{\tan}\frac{1}{2}\theta\right)$

or $e^{u} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

or $e^{\frac{u}{2}} \cdot e^{\frac{u}{2}} = \dfrac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4} \cdot \tan\frac{\theta}{2}}$

or $\dfrac{e^{\frac{u}{2}}}{e^{-\frac{u}{2}}} = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$

Applying componendo and dividendo we have

$$\frac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \frac{\left(1 + \tan\frac{\theta}{2}\right) - \left(1 - \tan\frac{\theta}{2}\right)}{\left(1 + \tan\frac{\theta}{2}\right) + \left(1 - \tan\frac{\theta}{2}\right)}$$

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or $\dfrac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \dfrac{\cancel{1} + \tan\frac{\theta}{2} - \cancel{1} + \tan\frac{\theta}{2}}{\cancel{1 + \tan\frac{\theta}{2}} + 1 - \cancel{\tan\frac{\theta}{2}}}$

or $\dfrac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \dfrac{\cancel{2}\tan\frac{\theta}{2}}{\cancel{2}}$

or $\dfrac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \tan\frac{\theta}{2}$

$\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Hence proved

(b) $\cosh u = \dfrac{\cosh^2\left(\frac{u}{2}\right) + \sinh^2\left(\frac{u}{2}\right)}{\cosh^2\left(\frac{u}{2}\right) - \sinh^2\left(\frac{u}{2}\right)}$

$$\cosh u = \dfrac{\dfrac{\cosh^2\left(\frac{u}{2}\right) + \sinh^2\left(\frac{u}{2}\right)}{\cosh^2\left(\frac{u}{2}\right)}}{\dfrac{\cosh^2\left(\frac{u}{2}\right) - \sinh^2\left(\frac{u}{2}\right)}{\cosh^2\left(\frac{u}{2}\right)}}$$

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$$= \frac{\frac{\cancel{\cosh^2(\frac{u}{2})}}{\cancel{\cosh^2(\frac{u}{2})}} + \frac{\sinh^2(\frac{u}{2})}{\cosh^2(\frac{u}{2})}}{\frac{\cancel{\cosh^2(\frac{u}{2})}}{\cancel{\cosh^2(\frac{u}{2})}} - \frac{\sinh^2(\frac{u}{2})}{\cosh^2(\frac{u}{2})}}$$ [dividing each term by $\cosh^2\frac{u}{2}$]

$$= \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$ [$\because \tanh(\frac{u}{2}) = \tan(\frac{\theta}{2})$]

$$= \frac{1}{\cos\theta}$$ [$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$]

$\cosh u = \sec\theta$

Hence proved

(c) $\sinh u$

$$= \frac{2\sinh(\frac{u}{2})\cosh(\frac{u}{2})}{\cosh^2(\frac{u}{2}) - \sinh^2(\frac{u}{2})}$$

$$= \frac{\frac{2\sinh(\frac{u}{2})\cosh(\frac{u}{2})}{\cos^2h(\frac{u}{2})}}{\frac{\cosh^2(\frac{u}{2}) - \sinh^2(\frac{u}{2})}{\cos^2h(\frac{u}{2})}}$$

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$$= \frac{\frac{2\sinh(\frac{u}{2})\cancel{\cosh}(\frac{u}{2})}{\cos^2\cancel{h}(\frac{u}{2})}}{\frac{\cosh^2(\cancel{\frac{u}{2}})}{\cosh^2(\frac{u}{2})} - \frac{\sinh^2(\frac{u}{2})}{\cosh^2(\frac{u}{2})}}$$

$$= \frac{\frac{2\sinh(\frac{u}{2})}{\cosh(\frac{u}{2})}}{1 - \frac{\sinh^2(\frac{u}{2})}{\cosh^2(\frac{u}{2})}}$$

$$= \frac{2\tanh(\frac{u}{2})}{1-\tanh^2(\frac{u}{2})}$$

$$= \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$\left[\tanh(\frac{u}{2}) = \tan(\frac{\theta}{2})\right.$

$$= \tan\theta$$

$\sinh u = \tan\theta$ proved

(d) $\tanh u$

from parts (b) and (c) we have

$\cosh u = \sec\theta$ and $\sinh u = \tan\theta$

Dividing we have

$$\frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta}$$

or $$\tanh u = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

or $$\tanh u = \frac{\sin\theta}{\cos\theta} \times \frac{\cos\theta}{1}$$

$$\tanh u = \sin\theta$$

proved