# Trigonometry

Page No.: Date: / /

## Hyperbolic functions

Hyperbolic functions: Whether $x$ be real or complex, the quantity $\frac{1}{2}(e^x - e^{-x})$ is defined as the hyperbolic sin $x$ and is written as $\sin hx$. And the quantity $\frac{1}{2}(e^x + e^{-x})$ is defined as the hyperbolic cosine of $x$ and is written as $\cos hx$

i.e, We write $\sin hx = \frac{e^x - e^{-x}}{2}$, $\cos hx = \frac{e^x + e^{-x}}{2}$

$$\tan hx = \frac{\sin hx}{\cos hx}$$

$$= \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$$

$$= \frac{e^x - e^{-x}}{\cancel{2}} \times \frac{\cancel{2}}{e^x + e^{-x}}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cot hx = \frac{\cos hx}{\sin hx}$$

$$= \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}}$$

$$= \frac{e^x + e^{-x}}{\cancel{2}} \times \frac{\cancel{2}}{e^x - e^{-x}}$$

$$= \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Some Important Results on Hyperbolic functions

(a) $\cosh^2\theta - \sinh^2\theta = 1$

proof :-

L.H.S $= \cosh^2\theta - \sinh^2\theta$

$$= \left[\frac{1}{2}(e^\theta + e^{-\theta})\right]^2 - \left[\frac{1}{2}(e^\theta - e^{-\theta})\right]^2$$

$$= \left[\frac{1}{4}(e^\theta + e^{-\theta})^2\right] - \left[\frac{1}{4}(e^\theta - e^{-\theta})^2\right]$$

$$= \frac{1}{4}\left[(e^\theta + e^{-\theta})^2 - (e^\theta - e^{-\theta})^2\right]$$

$$= \frac{1}{4}\left[(e^{2\theta} + 2e^\theta \times e^{-\theta} + e^{-2\theta}) - (e^{2\theta} - 2e^\theta \times e^{-\theta} + e^{-2\theta})\right]$$

$$= \frac{1}{4}\left[(\cancel{e^{2\theta}} + 2e^\theta \times e^{-\theta} + \cancel{e^{-2\theta}} - \cancel{e^{2\theta}} + 2e^\theta \times e^{-\theta} - \cancel{e^{-2\theta}})\right]$$

$$= \frac{1}{4}\left[2e^\theta \times e^{-\theta} + 2e^\theta \times e^{-\theta}\right]$$

Page No.:
Date: / /

$$= \frac{1}{4}\left[4e^{\theta} \times e^{-\theta}\right]$$

$$= \frac{1}{4}\left[4\cancel{e^{\theta}} \times \frac{1}{\cancel{e^{\theta}}}\right]$$

$$= \frac{1}{4}\left[4\right]$$

$$= \frac{1}{\cancel{4}} \times \cancel{4}$$

$$= 1$$

$= R.H.S$

(b) $\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta$

$$L.H.S = \cosh 3\theta$$
$$= \cos(3i\theta)$$
$$= 4\cos^3(i\theta) - 3\cos(i\theta)$$
$$= 4\cosh^3\theta - 3\cosh\theta \quad \left[\because \cos(i\theta) = \cosh\theta\right]$$

(c) $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$

$$L.H.S = \sinh 3\theta$$
$$= \frac{1}{i}\sin(3i\theta)$$
$$= \frac{1}{i}\left[3\sin(i\theta) - 4\sin^3(i\theta)\right]$$

Page No.:
Date : / /

$$= \frac{1}{(i)}\left[3(i\sinh\theta) - 4(i\sinh\theta)^3\right]$$

$$\left[\because \sin(i\theta) = i\sinh\theta\right]$$

$$= \frac{1}{(i)}\left[3i\sinh\theta - 4i^3\sinh^3\theta\right]$$

$$= \frac{1}{(i)}\left[3i\sinh\theta - 4\times i^2\times i\sinh^3\theta\right]$$

$$= \frac{1}{(i)}\left[3i\sinh\theta + 4i\sinh^3\theta\right]$$

$$\left[\because i^2 = -1\right]$$

$$= \frac{1}{\cancel{i}}\times 3\cancel{i}\sinh\theta + \frac{1}{\cancel{i}}\times 4\cancel{i}\sinh^3\theta$$

$$= 3\sinh\theta + 4\sinh^3\theta$$

$= R.H.S$

Page No.:
Date : / /

(d) Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$L.H.S = \cosh(x+y)$

$= \cos\{i(x+y)\}$ $[\because \cos(i\theta) = \cosh\theta]$

$= \cos(ix + iy)$

$= \cos(ix)\cos(iy) - (i\sin(ix))(i\sin(iy))$

$= \cosh x \cosh y + \sinh x \sinh y$

$= R.H.S$ proved $\left[\begin{array}{l}\cos(i\theta) = \cosh\theta \\ \sin(i\theta) = i\sinh\theta\end{array}\right]$

(e) $\sinh 2\theta = 2\sinh\theta\cosh\theta$

$L.H.S = \sinh 2\theta$

$= \left(\frac{1}{i}\right)\sin(2i\theta)$

$= \left(\frac{1}{i}\right) 2\sin(i\theta)\cos(i\theta)$ $\left[\begin{array}{l}\cos(i\theta) = \cosh\theta \\ \sin(i\theta) = i\sinh\theta\end{array}\right]$

$= \left(\frac{1}{i}\right) 2(i\sinh\theta)(\cosh\theta)$

$= \frac{1}{\cancel{i}} \times 2(\cancel{i}\sinh\theta)(\cosh\theta)$

$= 2\sinh\theta\cosh\theta$

$= R.H.S$ proved

These hyperbolic functions can be expressed in terms of circular functions as follows:

$$\sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$$

$$= \frac{e^{i^2x} - e^{-i^2x}}{2i} \quad [i^2 = -1]$$

$$= \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{-e^{x} + e^{-x}}{2i}$$

$$= -\frac{(e^{x} - e^{-x})}{2i}$$

$$= -\frac{1}{2i} \cdot (e^{x} - e^{-x})$$

$$= \frac{i^2}{2i}(e^{x} - e^{-x})$$

$$= \frac{i}{2}(e^{x} - e^{-x})$$

$$= i\,\frac{(e^{x} - e^{-x})}{2}$$

or

$$\sin(ix) = i \sinh x$$

Similarly $\cos ix = \dfrac{e^{i(ix)} + e^{-i(ix)}}{2}$

$$= \frac{e^{i^2x} + e^{-i^2x}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2} \quad [i^2 = -1]$$

or

$$\cos(ix) = \cosh x$$

$\therefore \tan(ix) = \dfrac{\sin ix}{\cos ix}$

$$= \frac{i \sinh x}{\cosh x}$$

or $\tan(ix) = i \tanh x$

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